Chronologically-ordered Rationalisable Choice

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Abstract

We propose a framework for the analysis of choice behaviour when the latter is made explicitly in chronological order. We relate this framework to the traditional choice theoretic setting from which the chronological aspect is absent. We show that for the analysis of traditional rational choice behaviour, introducing a chronological order at which choices take place is only relevant if a decision maker may face the same menu more than once. We then show that introducing a chronological order is crucial for analysing other plausible choice models. We illustrate this by proposing two models of behaviour in such a framework: (i) revealed changing preferences, and (ii) revealed preference discovery through trial and error. We provide a full characterization of these choice models by means of revealed preference-like axioms that could not be formulated in a timeless setting.

Keywords: Choice behaviour; Chronology; Time; Revealed preferences; Changing tastes; Preference discovery.

1 Introduction

An important accomplishment of modern economic theory is the precise identification of its behavioural implications. A rich and now classical tradition of research, initiated by Samuelson (1938) and pursued by Houthakker (1950), Chernoff (1954), Arrow (1959), Richter (1966), Sen (1971), among many others, has formulated these implications in terms of a choice function (sometimes generalized to a choice correspondence), that assigns to every set of alternatives (or menu) in some universe a unique element of it, interpreted as the chosen alternative in that menu. The behavioural implications of a significant variety of theories have been examined through the formalism of choice functions. The most well-known of them assumes that choices result from the maximization of a single preference defined on the set of all conceivable alternatives. The

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behavioural implications of this theory are Chernoff (1954) condition (called property $\alpha$ by Sen 1971), Houthakker’s (1950) Strong Axiom of Revealed Preference ($SARP$) or Richter’s (1966) congruence axioms.

The findings of psychology and behavioural economics suggest, however, that the observational implications of the maximization of a single preference are often rejected by actual choice behaviour (see Fehr and Hoff 2011, Kahneman 2011, Rabin 2013, and Hoff and Stiglitz 2016 for reviews). This has led several authors to propose alternative theories of choice and to look for the implications of these on choice functions. For example, Masatlioglu, Nakajima, and Ozbay (2012) have identified the properties of a choice function that selects the preferred alternative from a consideration set in each menu, rather than from the whole menu itself. This consideration set is interpreted as reflecting what the decision maker pays attention to. This consideration set may not coincide with the whole menu of feasible alternatives if, for example, the decision maker is “inattentive” to some of the alternatives that are available. Gerasimou (2017) has also used a choice-theoretic setting to characterize three models in which the decision maker either chooses the most preferred feasible option or “defers choice” (modelled as opting for an exogenous outside option) due to indecisiveness between various feasible options, the unattractiveness of these options or choice overload. Other papers, notably Manzini and Mariotti (2007, 2012), Mandler, Manzini, and Mariotti (2012), and Apesteguia and Ballester (2013), have identified the observable properties of a choice function that are necessary and sufficient for its rationalization by sequential lexicographic applications of a collection of preferences.

Flexible and amenable to formulations of testable implications of many behavioural choice models as it is, a choice function may still be considered unduly abstract for many applications. One of the important and easily observable features of reality that it neglects is the (time) period at which the menu is made available to the decision maker. Indeed, as used in the literature just described, a choice function describes a timeless process that only specifies the chosen alternatives in every admissible menu. It does not record (nor use information on) the periods at which the menus are available. Yet, in most choice data that we can think of, menus of alternatives present to the decision maker one after the other, and this chronological information is known. For instance, firms often record the time at which their consumers made their purchases, and economic experiments record the chronological order of choices made by their subjects. The data on the time at which choices are made is particularly relevant for choices — as discussed by Hoff and Stiglitz (2016) — that appear to be endogenous to context and experience.

In this paper, we extend the traditional choice-theoretic setting to analyse choice behaviour as an explicit function of both the period at which the choice takes place and the menu available at that period. We relate our framework to the classical model of choice correspondence and show that for the analysis of traditional rational choice behaviour, introducing a chronological order at which choices take place is only relevant if a decision maker may face the same menu more than once. We then show that this chronological data enables one
to identify the behavioural implications of alternative theories of choice that could not be analysed without an explicit integration of this information. We characterize in particular two such models.

The first is a model of *revealed changing preferences*, in which a decision maker chooses as if she changes preferences over time. In such a model, the decision maker chooses in a way that maximizes a given preference up to some time period and, after this period, switches to another preference and makes its subsequent choices based on this preference. We provide a simple “revealed preference test” for this particular theory of choice, that relates to the literature on changing tastes (see e.g. Gul and Pesendorfer 2005). While much of our discussion is focused on a choice model in which a decision maker changes preferences only once, we also generalize the model to allow the decision maker to change preferences any number of times that is strictly smaller than the total number of periods.\footnote{Trivially, any choice behaviour that depends upon time can be seen as resulting from a decision maker who changes preferences at every period. See Kalai, Rubinstein, and Spiegler (2002) for a similar observation on the standard timeless setting.} In this latter case, the characterization could be used to test how many changes in preferences are necessary to rationalize the observed behaviour of an individual.

The second is a model of *revealed preference discovery*, in which a decision maker chooses as if she discovers her preferences over two alternatives only after having “tried” them. In such a theory, the decision maker “tries out” the alternatives before forming her preference over them. Hence, when facing a menu at a given period, the decision maker either tries out one alternative or chooses the “best” option according to a single preference relation among the alternatives she has previously chosen. Just like for the previous model, we provide a full characterization when the decision maker discovers her preferences after one trial, but we also extend it to the case where it takes \( k \) trials to form a “definite” preference. This latter characterization could be used as an observational test to the “preference discovery hypothesis” formulated by Plott (1996), according to which decision makers behave inconsistently when first faced with unfamiliar tasks but with “sufficient” practice discover stable preferences (see also Smith 1994, Binmore 1999, Pierrmont, Takeoka, and Teker 2016, and Cooke 2017).

Our framework bears some formal similarity with that introduced by Bernheim and Rangel (2007, 2009) and Salant and Rubinstein (2008). These authors have analysed some normative (Bernheim and Rangel 2007, 2009) and positive (Salant and Rubinstein 2008) implications of choice processes in which every menu of alternatives is supplemented with an *ancillary condition* that represents either a “frame” or some other “normatively-irrelevant” feature of the choice environment. One could, at least in principle, view the time at which the choice is made as a frame or an ancillary condition. Yet time is a somewhat specific feature of the choice environment. One of its specificities is that it leads to a (chronological) ordering of the menus offered to the decision maker. The properties of this ordering (e.g. the fact that one alternative chosen “in the past” is not chosen “in the present”) play an important role in the characterization of the choice models described above. By contrast, the abstract ancillary
conditions and frames examined by Bernheim and Rangel (2007, 2009) and Salant and Rubinstein (2008) do not impose a structure on the set of available menus that is similar to that of an ordering. Another difference is that we do not assume the possibility of observing choice in any conceivable combination of time period and menu at that period. We only consider, somewhat realistically, that we observe a particular chronology of choices, and we identify the necessary and sufficient conditions that the choice behaviour observed in that particular chronology must satisfy to result from each of the different theories of choice mentioned above. A third difference is that the chronological order of subsets seems to deserve a different interpretation to that given to ancillary conditions or frames by Bernheim and Rangel (2007, 2009) and Salant and Rubinstein (2008). From a normative perspective, Bernheim and Rangel (2009) define an ancillary condition to be “a feature of the choice environment that may affect behaviour, but [that] is not taken as relevant to a social planner’s evaluation” (p. 55). As our analysis is more positive than normative, we do not take a position on this issue. Let us simply say that our intuition about this matter is that time should be relevant. After all, preferences (choices) that were revealed (made) very long time ago may have less bearing on our appraisal of the current well-being of the decision maker than those revealed (made) in more recent periods (particularly if past choices are no longer judged of worth or important; see Parfit 1984, ch. 8 for a related argument). But nothing in our analysis depends on this intuition. From a positive perspective, we also have difficulty in viewing the period at which a choice is made as a piece of information that is, to take Salant and Rubinstein (2008)’s words, “irrelevant in the rational assessment of the alternatives but nonetheless affects behaviour”. We conjecture that Salant and Rubinstein (2008), who do not give time as an example of a frame, would agree with us.

Finally, to the best of our knowledge Cerigioni (2017) is the only other paper that has — independently — used an analogous framework of chronological choice. However, while we focus on the characterization of individual behaviour by means of revealed-preference axioms, Cerigioni (2017) is interested on the probabilities of choosing alternatives that can be inferred from the choices of a collection of decision makers. In a different vein, Cerigioni (2016) proposes a choice theoretic framework that combines a chronology of choices with ancillary conditions, in which the menu available for choice at every period is supplemented by an abstract vector of (non-time) ancillary conditions that apply to that period. He characterizes in this framework a “dual-self” theory of choice. As compared to his, our analysis is closer to the classical choice theory since, except for a chronological order of menus, we do not consider any other argument of the choice function than the menu of alternatives to which it applies.

The rest of the paper proceeds as follows. In Section 2, we introduce our framework to study chronological choice. In Section 3, we characterize traditional rational choice in this setting and show that for the analysis of this type of behaviour, introducing a chronological order at which choices take place is only relevant if a decision maker may face the same menu more than once. In Section 4, we propose and characterize our model of revealed changing preferences,
and in Section 5 we propose and characterize our model of revealed preference discovery. Section 6 concludes.

2 The framework

2.1 General notation

In the following, we define a binary relation $\succeq$ on any set $\Omega$ as a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succeq y$ instead of $(x, y) \in \succeq$. Given a binary relation $\succeq$, we define its symmetric factor $\sim$ by $x \sim y$ if $x \succeq y$ and $y \succeq x$ and its asymmetric factor $\succ$ by $x \succ y$ if $x \succeq y$ and not $(y \succeq x)$.

A binary relation $\succeq$ on $\Omega$ is:

(i) reflexive if the statement $x \succeq x$ holds for every $x$ in $\Omega$,

(ii) transitive if $x \succeq z$ follows $x \succeq y$ and $y \succeq z$ for any $x, y, z \in \Omega$,

(iii) complete if $x \succeq y$ or $y \succeq x$ holds for every distinct $x$ and $y$ in $\Omega$ and,

(iv) antisymmetric if $x \sim y \Rightarrow x = y$.

We call ordering a reflexive, transitive, and complete binary relation and linear ordering an antisymmetric ordering. Given two binary relations $\succeq_1$ and $\succeq_2$ we say that $\succeq_2$ is an extension of $\succeq_1$ (or is compatible with $\succeq_1$) if it is the case that, for any $x$ and $y$ in $\Omega$ such that $x \succeq_1 y$ one has also $x \succeq_2 y$.

Given a binary relation $\succeq$ on a set $\Omega$, we define its transitive closure $\succeq^*$ by: $x \succeq^* y \iff \exists\{x_j\}_{j=0}^l$ for some integer $l \geq 1$ satisfying $x_j \in \Omega$ for all $j = 0, ..., l$ for which one has $x_0 = x$, $x_l = y$ and $x_j \succeq x_{j+1}$ for all $j = 0, ..., l - 1$. It is well-known that the transitive closure of a binary relation $\succeq$ is the smallest (with respect to set inclusion) transitive binary relation compatible with $\succeq$. We also denote by $\mathbb{N}$ and $\mathbb{N}_+$ the set of non-negative and strictly positive integers (respectively), and by $\#A$ the cardinality of any finite set $A$.

2.2 Modelling chronological choice

Let $X$ be a universe of alternatives of interest for the decision maker, $\mathcal{P}(X)$ be the set of all finite and non-empty subsets of $X$, and $\mathcal{F}$ be a collection of non-empty finite subsets of $X$, each of which is interpreted as a possible menu that can be offered to a decision maker. A choice correspondence on $\mathcal{F}$ is a mapping $C : \mathcal{F} \rightarrow \mathcal{P}(X)$ that satisfies $C(A) \subset A$ for every $A$ in $\mathcal{F}$. A choice correspondence on $\mathcal{F}$ that has the property that $\#C(A) = 1$ for every $A$ in $\mathcal{F}$ is called a choice function. Choice functions (correspondences) provide abstract descriptions of a choice process in which a decision maker facing various menus of alternatives chooses one (some) alternative(s) in each menu. The choice-theoretic literature that has emerged in the last sixty years or so has made various assumptions on the domain $\mathcal{F}$ that depend, sometimes, upon the nature of the alternatives in $X$ that are considered. For example, the classical theory introduced by Arrow (1959) has taken $X$ to be an abstract finite set, and $\mathcal{F}$ to coincide with the set of all non-empty subsets of $X$. This is clearly demanding from an observational viewpoint, since it is difficult in practice to
observe all choices that an agent could make in every logically conceivable menu. However, in subsequent years, several authors (including Richter 1966, Hansson 1968, and Suzumura 1976, 1977, 1983) have shown that several results on the rationalization of a choice function by a single preference hold on “general” domains that do not impose any restriction on the class of menus that may be available.

In this paper, we supplement this general setting with a discrete time horizon that enables the definition of a chronology of choices as a list of pairs \( \{A^t, a^t\}_{t=1}^T \) for some (strictly positive integer) \( T \) where, for every period \( t, A^t \in \mathcal{P}(X) \) and \( a^t \in A^t \). The interpretation is that, at every period \( t \), a particular menu \( A^t \) of alternatives is presented to the decision maker who chooses the alternative \( a^t \) in that menu. In this setting, it is possible that, for two different periods \( t \) and \( t' \), \( a^t \neq a^{t'} \) even though \( A^t = A^{t'} \). That is, a decision maker who faces the same menu at two different periods may make different choices from this menu.

It is worth noting that the data used in a chronology of choices abstracts from situations in which the decision maker’s choice in one period affects the menus of alternatives that will be available in future periods. Due to this, our framework is not adapted to handle many important classical issues pertaining to dynamic and forward-looking behaviour such as time consistency, myopia, and preferences for flexibility (see e.g. Strotz 1955, Koopmans 1964, Peleg and Yaari 1973, Hammond 1976 and Kreps 1979). It seems instead best suited for exogenous and unfamiliar decisions/tasks or other circumstances in which the decision maker has no expectations about future menus.

Just like in the standard timeless framework, the behavioural implications of the decision-making models we will look at will take the form of axioms that are formulated in terms of “revealed preference” relations. Two types of such relations will play an important role in the formulation of the axioms. The first one is the direct revealed preference relation at period \( s \) that is defined as follows.

**Definition 1** Given a chronology of choices \( \{A^t, a^t\}_{t=1}^T \) and a period \( s \in \{1, \ldots, T\} \), we say that \( x \) is **directly revealed preferred to** \( y \) at period \( s \), denoted \( x \succsim^s y \), if and only if \( x = a^s \) and \( y \in A^s \).

In words, \( x \) is directly revealed preferred to \( y \) at period \( s \) whenever \( x \) is chosen at period \( s \) and \( y \) is available at that period. This direct revealed preference at period \( s \) is analogous to the notion formulated by Arrow (1959) in a timeless setting. In the spirit now of Houthakker (1950), one can define the notion of indirect revealed preference relation between periods \( r \) and \( s \) as follows:

**Definition 2** Given a chronology of choices \( \{A^t, a^t\}_{t=1}^T \) and any periods \( r \) and \( s \in \{1, \ldots, T\} \) such that \( r \leq s \), we say that \( x \) is **indirectly revealed preferred to** \( y \) between periods \( r \) and \( s \), denoted \( x \succsim^{rs} y \), if and only if there is a sequence \( \{t_j\}_{j=1}^k \) of \( k \) time periods in the set \( \{r, r+1, \ldots, s-1, s\} \), for which one has:
We observe that, by the very definition of a chronology of choices, the binary relation $\succsim$ is antisymmetric for every period $s$. However, for an arbitrary pair of periods $r$ and $s$ satisfying $r \leq s$, the binary relation $\succsim_{rs}$ need not be antisymmetric. The fact of having $x \succsim_{rs} y$ for two distinct alternatives $x$ and $y$ does not preclude the possibility of having $y \succsim_{rs} x$. We also emphasize that the sequence of sets involved in the definition of the indirect revealed preference between periods $r$ and $s$ need not be chronologically ordered. For example, suppose that $X = \{a, b, c, d, e\}$, $r = 1$, $s = 3$ and that the chronology of choices for the periods 1, 2 and 3 is:

\[
\begin{align*}
\{A^1, a^1\} &= \{\{a, b, c\}, a\} \\
\{A^2, a^2\} &= \{\{d, a\}, d\} \text{ and} \\
\{A^3, a^3\} &= \{\{b, c\}, b\}.
\end{align*}
\]

It follows from Definition 2 that alternative $d$ is indirectly revealed preferred to alternative $e$ between periods 1 and 3. In effect, $d$ has been directly revealed preferred to $a$ in period 2, which has been itself directly revealed preferred to $b$ in period 1, which has been directly revealed preferred to $e$ at period 3. The sequence of direct revealed preference statements that connect $d$ to $e$ between periods 1 and 3 is therefore not indexed by time.

Finally, note that one can define for any chronology of choices $\{A^t, a^t\}^T_{t=1}$, an induced timeless family of menus $\mathcal{F}(\{A^t, a^t\}^T_{t=1}) = \{A \in \mathcal{P}(X) : \exists s \in \{1, ..., T\} \text{ such that } A = A^s\}$. This family $\mathcal{F}(\{A^t, a^t\}^T_{t=1})$ is the set of all menus faced by the decision maker in the chronology of choices $\{A^t, a^t\}^T_{t=1}$ when “abstracting” from the time period at which the menus appear. Then, it is possible to define for any chronology of choices $\{A^t, a^t\}^T_{t=1}$ a timeless choice correspondence $C : \mathcal{F}(\{A^t, a^t\}^T_{t=1}) \to \mathcal{P}(X)$ such that $C(A) = \{x \mid x = a^t \text{ for some } t \in \{1, ..., T\} \text{ such that } A^t = A\}$. Note that this timeless choice correspondence $C$ will actually be a choice function if no menu is proposed to the decision maker more than once. This formal link to the classical model of choice correspondence will be used in the next section when we analyse traditional rational choice behaviour in our setting.

3 Chronological rational choice

Standard choice theory assumes that decision makers make their choices by maximizing a time-invariant ordering. While empirical evidence, casual observation, and introspection suggest that this assumption is not always realistic, it does represent a sensible benchmark in many situations (see e.g. Roth 1996, Plott 1996, Loewenstein 1999, Loomes 1999 and Starmer 1999). We, therefore, start our analysis by characterizing a chronology of choices resulting from the maximization of a (time-invariant) single linear preference:
Definition 3 A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) results from the maximization of a single preference if and only if there exists a linear ordering \( \succeq \) on \( X \) such that, for every \( t \in \{1, ..., T\} \), \( a^t \succeq a' \) for all \( a' \in A^t \).

The observable property of the chronology of choices that characterizes this behaviour is the following version of SARP applied to a chronological setting:

Axiom 1 (T-SARP) A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) satisfies T-SARP if, for any periods \( r, s \) and \( t \in \{1, ..., T\} \) such that \( r \leq s < t \) and some distinct \( x \) and \( y \in X \), one cannot have \( x \succeq^{rs} y \) and \( y \succ^t x \).

We now establish that T-SARP is necessary and sufficient for a chronology of choices to result from the maximization of a single time-invariant preference.

Theorem 1 A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) satisfies T-SARP if and only if it results from the maximization of a single preference.

Proof. See Appendix (for all proofs).

We now relate our framework to the classical model of choice correspondence, and show that for the analysis of traditional rational choice behaviour introducing a chronological order at which choices take place is only relevant if a decision maker may face the same menu more than once. To see this, we first recall the following definitions that apply to a general choice correspondence \( C : \mathcal{F} \rightarrow \mathcal{P}(X) \).

Definition 4 (Weak direct revealed preference) For any two alternatives \( x \) and \( y \in X \), we say that the choice correspondence reveals a weak direct preference for \( x \) over \( y \), denoted \( x \succeq_D y \), if there exists a menu \( A \in \mathcal{F} \) such that \( x \in C(A) \) and \( y \in A \).

Definition 5 (Strict direct revealed preference) For any two alternatives \( x \) and \( y \in X \), we say that the choice correspondence reveals a strict direct preference for \( x \) over \( y \), denoted \( x \succ_D y \), if there exists a menu \( A \in \mathcal{F} \) such that \( x \in C(A) \) and \( y \in A \setminus C(A) \).

Definition 6 (Weak Indirect revealed preference) For any two alternatives \( x \) and \( y \in X \), we say that the choice correspondence reveals an indirect preference for \( x \) over \( y \), denoted \( x \succeq_I y \), if there exists a sequence of \( j \) alternatives \( \{z_j\}_{j=1}^{\bar{j}} \) (for some \( \bar{j} \in \mathbb{N}_+ \)) satisfying \( z_j \neq z_{j+1} \) for all \( j = 1, ..., \bar{j}-1 \) such that:

(i) \( z_1 = x \)
(ii) \( z_j \succeq_D z_{j+1} \) for all \( j = 1, ..., \bar{j}-1 \) and,
(iii) \( z_{\bar{j}} = y \)

Axiom 2 (SARP) We say that the choice correspondence \( C : \mathcal{F} \rightarrow \mathcal{P}(X) \) satisfies SARP if for any two alternatives \( x \) and \( y \in X \), one can not have \( x \succeq_I y \) and \( y \succ_D x \).
We can now show that when a chronology of choices \( \{ A^t, a^t \}_{t=1}^T \) is such that 
\( A^t \neq A^{t'} \) for any two distinct periods \( t \) and \( t' \) — in which case the choice correspondence 
\( C : F(\{ A^t, a^t \}_{t=1}^T) \rightarrow P(X) \) is a function — there is no difference between requiring the chronology of choices 
\( \{ A^t, a^t \}_{t=1}^T \) to satisfy T-SARP and requiring the choice function 
\( C : F(\{ A^t, a^t \}_{t=1}^T) \rightarrow P(X) \) to satisfy SARP.

**Proposition 1** Let \( \{ A^t, a^t \}_{t=1}^T \) be a chronology of choices in which 
\( A^t \neq A^{t'} \) for any two distinct \( t, t' \in \{ 1, ..., T \} \). Then \( \{ A^t, a^t \}_{t=1}^T \) satisfies T-SARP if and only if 
\( C : F(\{ A^t, a^t \}_{t=1}^T) \rightarrow P(X) \) satisfies SARP.

In other words, when a decision maker is not confronted with the same menu more than once, there is nothing to be gained by introducing a chronology of choices for the analysis of traditional rational choice behaviour. On the other hand, Proposition 1 does not hold in the case of chronology of choices in which the decision maker may face the same menu at two different periods. To see this, consider the chronology of choices 
\( \{ A^t, a^t \}_{t=1}^T \) defined by:

\[
\begin{align*}
\{ A^1, a^1 \} &= \{ \{ a, b, c \}, a \} \\
\{ A^2, a^2 \} &= \{ \{ b, c \}, b \} \text{ and} \\
\{ A^3, a^3 \} &= \{ \{ a, b, c \}, b \}.
\end{align*}
\]

The family of sets generated by this chronology is 
\( F(\{ A^t, a^t \}_{t=1}^T) = \{ \{ a, b, c \}, \{ b, c \} \} \) and its choice correspondence \( C \) is defined by 
\( C(\{ a, b, c \}) = \{ a, b \} \) and \( C(\{ b, c \}) = \{ b \} \). It is clear that \( C \) satisfies SARP. However, the chronology of choices violates T-SARP because \( a \succeq^1 b \) and \( b \succeq^3 a \).

However, as shown in the next two sections, there are other plausible theories of choice behaviour whose characterization does depend crucially on the chronological order at which the choices are made.

**4 A model of revealed changing preferences**

In this section, we examine a model in which a decision maker behaves as if she changes preferences over time. We start with the simple (and plausible) case where the decision maker preferences change at most once. From the observer point of view, this corresponds to the case where there is a single period, unknown \textit{a priori} by the observer, in which the decision maker “switches” from one preference to another, as would a wine lover who has drunk too much wine yesterday and who may today reveal a preference for milk (Samuelson 1952, p. 674). To illustrate even further the model we have in mind, consider the following examples.

**Example 1** Consider the following chronology of choices:

\[
\begin{align*}
\{ A^1, a^1 \} &= \{ \{ \text{chicken, dahl} \}, \text{chicken} \} \\
\{ A^2, a^2 \} &= \{ \{ \text{chicken, dahl, tuna} \}, \text{chicken} \} \text{ and} \\
\{ A^3, a^3 \} &= \{ \{ \text{beef, chicken, dahl} \}, \text{dahl} \}.
\end{align*}
\]
This choice behaviour is consistent with one change in preferences. The decision maker behaves as if she switches once for a “vegetarian” preference between the second and the third periods. The timeless choice function associated to this chronology of choices is $C(\{\text{chicken, dahl}\}) = \text{chicken}$, $C(\{\text{chicken, dahl, tuna}\}) = \text{chicken}$ and $C(\{\text{beef, chicken, dahl}\}) = \text{dahl}$, which clearly violates SARP.

Example 2 Consider the following chronology of choices:

$\{A_1, a_1\} = \{\{\text{chicken, dahl}\}, \text{chicken}\}$
$\{A_2, a_2\} = \{\{\text{chicken, dahl, tuna}\}, \text{dahl}\}$ and
$\{A_3, a_3\} = \{\{\text{beef, chicken, dahl}\}, \text{chicken}\}$.

This choice behaviour is not consistent with one change in preferences. While the choices at the first and at the last period reveal a preference for chicken over dahl, the choice made at the second period reveals a preference for dahl over chicken. To generate such a pattern of choices, the decision maker must have changed preferences at every period (i.e. twice). From the viewpoint of conventional choice theory, the choice function induced by this chronology of choices is $C(\{\text{chicken, dahl}\}) = \text{chicken}$, $C(\{\text{chicken, dahl, tuna}\}) = \text{dahl}$ and $C(\{\text{beef, chicken, dahl}\}) = \text{chicken}$. Again, it clearly violates SARP.

Although choices in Example 1 are consistent with one change in preferences and choices in Example 2 are not, both timeless choice functions associated with these chronologies of choices violate SARP. It is therefore not possible to distinguish the choice behaviour depicted in these examples using traditional revealed-preference axioms, while it is possible with the explicit introduction of a chronological order.

We now define what is meant by a chronology of choices to result from at most one change in preferences:

Definition 7 A chronology of choices $\{A_t, a_t\}_{t=1}^{T}$ results from at most one change in preferences if there exist two linear orderings $\succsim_1$ and $\succsim_2$ on $X$ and a period $t \in \{1, \ldots, T\}$ such that, for all periods $j, v \in \{1, \ldots, T\}$ such that $j < t \leq v$, one has $a_j \succsim_1 a'_j$ for all $a'_j \in A_j$ and $a_v \succsim_2 a'_v$ for all $a'_v \in A_v$.

The following simple property characterizes a chronology of choices that results from at most one change in preferences:

Axiom 3 (T-SARP after 1 change) A chronology of choices $\{A_t, a_t\}_{t=1}^{T}$ satisfies T-SARP after 1 change if, for any periods $r, s$ and $t \in \{1, \ldots, T\}$ such that $r \leq s < t$ and distinct $x$ and $y \in X$, if $x \succsim_{r}^{s} y$ and $y \succsim_{s}^{r} x$, then one can not have $w \succsim_{\sigma}^{u} z$ and $z \succsim_{\tau}^{v} w$ for any two distinct alternatives $w$ and $z \in X$ and periods $u, v$ and $\tau$ such that $t \leq u \leq v < \tau$.

This axiom says that if one observes a violation of T-SARP between period 1 and a given period $t$, then it is not possible to observe a second violation of T-SARP between $t$ and $T$. This axiom is therefore almost as easy to test as T-SARP. The following theorem formally states the equivalence between Definition 7 and the property T-SARP after 1 change.
Theorem 2 A chronology of choices \( \{A^t, a^t\}_{t=1}^T \) satisfies T-SARP after 1 change if and only if it results from at most one change in preferences.

Theorem 2 hence provides an easy way to test if the behaviour of an agent is consistent with at most one change in preferences, and choosing at each period in a way that maximizes the prevailing preference at that period. It is possible to extend this model to any number of preference changes that is strictly smaller than \( T \) (see Appendix A.2). This extension could be used to identify the minimal number of preference changes that are necessary to rationalize behaviour.

5 A model of revealed preference discovery

We now propose a revealed preference discovery model, in which the agent reveals (discovers) a “definite” preference between two alternatives only after having chosen (tried) them. At each period, an agent either chooses the “best” alternative among the ones she has already “tried” or tries a new alternative. The most immediate case is when the decision maker reveals a definite preference between two alternatives after having tried them once. For example, a foreigner in an exotic country is likely to “discover” her preference between local foods only after having tried them for the first time. This choice model is for the most part consistent with “rational behaviour”, but accommodates some learning that may lead to initial “inconsistencies” in choices.

We find again useful to consider two examples that demonstrate the reach of the model:

Example 3 Consider the following chronology of choices:

\[
\begin{align*}
{A^1, a^1} &= \{\text{beef, chicken, dahl}\}, \text{chicken} \\
{A^2, a^2} &= \{\text{beef, chicken}\}, \text{beef} \\
{A^3, a^3} &= \{\text{chicken, dahl}\}, \text{chicken} \text{ and} \\
{A^3, a^3} &= \{\text{beef, dahl}\}, \text{beef}.
\end{align*}
\]

This behaviour is consistent with a model of revealed preference discovery after one trial. Albeit one observes a violation of SARP in the traditional sense, this violation is interpreted as the result of trial and error. After trials in period 1 and 2, the decision maker reveals a “definite” preference for chicken over beef in period 3, and she is consistent with this preference in the following period.

Example 4 Consider the following chronology of choices:

\[
\begin{align*}
{A^1, a^1} &= \{\text{chicken, dahl}\}, \text{chicken} \\
{A^2, a^2} &= \{\text{beef, dahl}\}, \text{beef} \\
{A^3, a^3} &= \{\text{beef, chicken, dahl}\}, \text{chicken} \text{ and} \\
{A^3, a^3} &= \{\text{beef, chicken}\}, \text{beef}.
\end{align*}
\]
This choice behaviour is not consistent with a model of revealed preference discovery after one trial. According to the model, in the first period the decision maker chooses chicken and tries its taste, and in the second period she chooses beef and tries out its taste. In the third period, given that she knows the tastes, the choice reveals a “definite” preference for chicken over beef. However, the choice at the fourth period — beef over chicken — is inconsistent with this preference.

We emphasize, here again, the crucial importance of introducing a chronology for characterizing a behaviour resulting from preference discovery. Indeed, the only difference between the two examples is the chronological order at which the menus — identical in both examples — appear. Hence, the choice behaviours in these two examples induce exactly the same timeless choice function $C$ defined on the same family $F\{\{A^t, a^t\}_{t=1}^{T}\}$ and induce the same violation of SARP. However, choices in Example 3 are consistent with revealed preference discovery after one trial while choices in Example 4 are not.

To characterize this choice model, we redefine what is meant by direct revealed preference. More precisely, we define the “direct revealed definitely preferred” relation as follows:

**Definition 8** Given a chronology of choices $\{A^t, a^t\}_{t=1}^{T}$, a period $t \in \{1, ..., T\}$ and two distinct alternatives $x$ and $y \in X$, we say that $x$ is directly revealed definitely preferred to $y$ at period $t$, denoted $x \succ_d y$, if and only if there are periods $r$ and $s \in \{1, ..., T\}$ satisfying $r < t$ and $s < t$ such that $x = a^r = a^t$, $y \in A^t$ and $y = a^s$.

In words, the chronology of choices directly reveals a definite preference for $x$ over $y$ at period $t$ whenever $x$ is chosen over $y$ at a period $t$, under the condition that this period $t$ follows two periods where $x$ and $y$ have been chosen before. Given this relation, one defines the revealed definitely preferred relation over a sequence of periods going from $r$ up to $s$ as follows:

**Definition 9** Given a chronology of choices $\{A^t, a^t\}_{t=1}^{T}$, two periods $r$ and $s \in \{1, ..., T\}$ such that $r < s$ and two distinct alternatives $x$ and $y \in X$, we say that $x$ is indirectly revealed definitely preferred to $y$ between periods $r$ and $s$, denoted $x \succ_{rs} y$, if and only if there is a sequence $\{t_j\}_{j=1}^{k}$ of $k$ time periods in the set $\{r, r + 1, ..., s - 1, s\}$ for which one has:

(i) $x = a^{t_1}$,
(ii) $a^{t_j} \succ_{d} a^{t_{j+1}}$ for all $j = 1, ..., k - 1$, and
(iii) $y \in A^{t_k}$ and $y = a^q$ for some $q < t_k$.

Given these definitions, we can formally define what we mean by a chronological choice behaviour to result from the maximization of a discovered preference:

**Definition 10** A chronology of choices $\{A^t, a^t\}_{t=1}^{T}$ results from the maximization of a discovered preference after one trial if there exists a linear ordering $\succ$ on $X$ such that, for all $t \in \{1, ..., T\}$, either $a^t \succeq a^t_1$ for all $a^t_1 \in A^t$ for which $a^t_1 = a^s$ for some period $s < t$ or there is no period $r < t$ for which $a^t = a^r$. 

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That is, a chronological choice behaviour results from the maximization of a discovered preference if there exists a preference such that the choice made by the decision maker at every period is either the “best” option for that preference among all alternatives that have been previously tried once or, if this is not the case, the chosen option has not been tried before. Note that the “best” option for that preference among all alternatives that have been previously tried is not necessarily the maximal option for that preference. It is possible that the maximal option has not been tried before. Of course, in this case, the decision maker has not yet “discovered” that this option is maximal. The following property is necessary and sufficient for a chronology of choices to result from the maximization of a discovered preference.

**Axiom 4 (T-SARP after 1 trial)** A chronology of choices \( \{A^t, d^t\}_{t=1}^{T} \) satisfies T-SARP after 1 trial if, for any periods \( r, s, t \) such that \( r \leq s < t \) and some distinct \( x \) and \( y \in X \), one cannot have \( x \succsim^r_s y \) and \( y \succ^t d x \).

In plain English, this property requires the decision maker to be consistent in her choices when those choices concern alternatives that have been tried at least once before. We can then establish the following.

**Theorem 3** A chronology of choices satisfies T-SARP after 1 trial if and only if it results from the maximization of a discovered preference after one trial.

Theorem 3 provides an easy way to test if behaviour that is inconsistent with traditional rational choice is instead the result of a process of trial and error. It is straightforward to extend this theory to the case where \( k \)-trials are needed (for any exogenously given \( k < T \)) before forming a “definite” preference between alternatives (see Appendix A.3). This extension could be used to test Plott’s (1996) hypothesis that a decision maker needs “sufficient” repetition to discover one’s preferences over alternatives.

**6 Concluding remarks**

In this paper, we have argued in favour of explicitly introducing time in the description of choice behaviour provided by a choice function. First, we have shown that for the analysis of traditional rational choice behaviour, introducing a chronological order at which choices take place is only relevant if a decision maker may face the same menu more than once. Second, we have used our framework to characterize the observational implications of two plausible theories of choice that have been discussed in economics. Importantly, these theories could not be formulated and characterized without an explicit integration of a chronological order of choice situations. Our characterizations can be used to test competing explanations for inconsistent behaviour, and when extended to any number of changes of preferences (or trials), they can be used to test how many changes in preferences (or trials) are necessary to rationalize the behaviour of an individual.
We note that the behavioural implications in this paper were formulated in terms of indirect revealed preference relations. While this is quite standard in the choice theoretic literature, it may become computationally demanding to test them if the universe of alternatives is large. We also find worth pointing out the ease with which these characterizations were obtained. The simple fact of introducing time in the description of choice behaviour seems to have the significant payoff of alleviating what Rubinstein (2012, p. 40) calls the “burden on researchers” of finding the observable properties of the models that they are interested in.

Finally, a fruitful extension of this work would be to characterize similar models under stochastic choice. For example, in a stochastic choice setting one could model an agent that would progressively learn her preferences, as in the model of revealed preference discovery above, but she would commit some “errors” even after learning has stopped. We leave the extension to that setting for future work.

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References


Appendixes

A.1 Proofs of Theorems 1, 2, and 3 and Proposition 1

A.1.1 Proof of Theorem 1

We first show that a chronology of choices \( \{ A_t, a_t \}_{t=1}^T \) for which there exists a linear ordering \( \succeq \) on \( X \) such that, for every \( t \in \{ 1, ..., T \} \), one has \( x = a^t \) if and only if \( x \succeq x' \) for all \( x' \in A^t \) satisfies T-SARP. For this sake, assume the existence of a linear ordering \( \succeq \) on \( X \) such that, for every \( t \in \{ 1, ..., T \} \), one has \( x = a^t \) if and only if \( x \succeq x' \) for all \( x' \in A^t \) and consider any periods \( r, s \) and \( t \) such that \( r \leq s < t \) and some distinct \( x \) and \( y \in X \) for which we have \( x \succeq r y \). By Definition 2, there is a sequence \( \{ t_j \}_{j=1}^k \) of \( k \) time periods in the set \( \{ r, r+1, ..., s-1, s \} \) for which one has:

(i) \( x = a^t \),
(ii) \( a^{t_j} \succeq t_j a^{t_{j+1}} \) for all \( j = 0, ..., k-1 \) and,
(iii) \( y \in A^{t_k} \).

Since the chronology of choices \( \{ A^t, a^t \}_{t=1}^T \) is rationalized by the linear ordering \( \succeq \), one has \( a^{t_j} \succeq a^{t_{j+1}} \) for all \( j = 1, ..., k-1 \) and, therefore, \( x \succeq y \) by the transitivity of \( \succeq \).

To prove the other implication, consider a chronology of choices \( \{ A^t, a^t \}_{t=1}^T \) that satisfies T-SARP. Define the binary relation \( \succeq_C \) on \( X \) by:

\[ x \succeq_C y \iff \exists t \in \{ 1, ..., T \} \text{ s.t. } x = a^t \text{ and } y \in A^t. \]

Define also the binary relation \( \widehat{\succeq}_C \) by:

\[ x \widehat{\succeq}_C y \iff \exists \{ t_j \}_{j=0}^k \text{ with } t_j \in \{ 1, ..., T \} \text{ for } j = 0, ..., k \text{ and } k \geq 0 \text{ such that:}
\]

(i) \( x = a^{t_0} \),
(ii) \( a^{t_j+1} \in A^{t_j} \) for \( j = 0, ..., k-1 \) (if any)
(iii) \( y \in A^{t_k} \)

It is immediate to see that \( \widehat{\succeq}_C \) is the transitive closure of \( \succeq_C \) and is therefore transitive.

We now show that \( \widehat{\succeq}_C \) is antisymmetric if the chronology of choices \( \{ A^t, a^t \}_{t=1}^T \) satisfies T-SARP. By contradiction, suppose \( \widehat{\succeq}_C \) is such that there are two distinct alternatives \( x \) and \( y \in X \) for which both \( x \widehat{\succeq}_C y \) and \( y \widehat{\succeq}_C x \) hold. This means that:

\[ \exists \{ t_j \}_{j=0}^k \text{ with } t_j \in \{ 1, ..., T \} \text{ for } j = 0, ..., k \text{ (with } k \geq 0 \text{) such that:}
\]

(i) \( x = a^{t_0} \),
(ii) \( a^{t_j+1} \in A^{t_j} \) for \( j = 0, ..., k-1 \) (if any)
(iii) \( y \in A^{t_k} \)
and

$$\exists \{t'_j\}_{j=0}^{k'} \text{ with } t'_j \in \{1, \ldots, T\} \text{ for } j = 0, \ldots, k' \text{ (with } k' \geq 0 \text{) such that:}$$

(i) \( y = a'_{t'_0} \)

(ii) \( a'_{t'_j+1} \in A'_{t'_j} \) for \( j = 0, \ldots, k' - 1 \) (if any)

(iii) \( x \in A'_{t'_k} \)

(2)

Consider the sets of time periods \( \mathcal{T} = \bigcup_{j=0}^{k} \{t_j\} \) and \( \mathcal{T}' = \bigcup_{j=0}^{k'} \{t'_j\} \) involved in expressions (1) and (2) respectively. As these two expressions define a cycle of revealed preference relations connecting alternative \( x \) to itself (or any chosen element in the cycle to itself), this cycle can be started at any \( t \in \mathcal{T} \cup \mathcal{T}' \) that we wish. In particular, \( t \) can be the maximal (with respect to the natural ordering of time) such period in \( \mathcal{T} \cup \mathcal{T}' \). We then have \( a' \succ a^s \) for some \( s < t \). By definition of the cycle induced by expressions (1) and (2), there are also periods \( r \) and \( r' \in (\mathcal{T} \cup \mathcal{T}') \setminus \{t\} \) satisfying \( r \leq r' \) that produce the opposite indirect preference \( a^r \succ r' \succ a^s \). But this contradicts \( T\text{-SARP} \). Hence \( \succ_C \) is an antisymmetric and transitive binary relation. By Spilrajn extension theorem, one can therefore extend \( \succ_C \) into a complete linear ordering \( \succ \).

Let us now show that for every \( t \in \{1, \ldots, T\} \), one has \( x = a^t \iff x \succ_C a \) for all \( a \in A^t \). Consider indeed any \( t \in \{1, \ldots, T\} \). Assume first \( x = a^t \). Then, by definition of \( \succ_C \), one has \( x \succ_C a \) for every \( a \in A^t \) so that the implication \( x \succ_C a \) for every \( a \in A^t \) follows from the fact that \( \succ_C \) extends \( \succ \) which extends itself \( \succ_C \). Assume now that \( x \succ a \) for every \( a \in A^t \) for some \( x \in A^t \) and every \( t \in \{1, \ldots, T\} \). Suppose by contradiction that \( x \not\succ a^t \). Then, there exists some alternative \( y \) distinct from \( x \) such that \( y = a^t \). By definition of \( \succ_C \), one has \( y \succ_C x \) and, therefore, \( y \succ_C x \) and \( y \succ x \). But, since \( x \succ_C a \) for every \( a \in A^t \), this is incompatible with \( \succ_C \) being antisymmetric.

**A.1.2 Proof of Proposition 1**

Let \( \{A^t, a^t\}_{t=1}^{T} \) be a chronology of choices in which \( A^t \neq A^{t'} \) for all distinct \( t, t' \in \{1, \ldots, T\} \). Let us show that if the choice correspondence \( C : \mathcal{F}(\{A^t, a^t\}_{t=1}^{T}) \rightarrow \mathcal{P}(X) \) violates \( \text{SARP} \), then the chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) violates \( T\text{-SARP} \). Since \( C \) is a choice function if \( A^t \neq A^{t'} \) for any two distinct \( t, t' \in \{1, \ldots, T\} \), we write, for any \( A \in \mathcal{F}(\{A^t, a^t\}_{t=1}^{T}) \), \( x = C(A) \) instead of \( x \in C(A) \). Since \( C : \mathcal{F}(\{A^t, a^t\}_{t=1}^{T}) \rightarrow \mathcal{P}(X) \) violates \( \text{SARP} \), there are \( k+1 \) distinct sets \( A_0, \ldots, A_k \) in the family \( \mathcal{F}(\{A^t, a^t\}_{t=1}^{T}) \) (for some \( k \in \mathbb{N}_+ \)) and alternatives \( a_0, \ldots, a_k \) in \( X \) such that \( a_j = C(A_j) \) for all \( j = 0, \ldots, k \), \( a_{j+1} \in A_j \) for \( j = 0, \ldots, k-1 \) and \( a_0 \in A_k \setminus \{a_k\} \). Since \( A^t \neq A^{t'} \) for all distinct \( t, t' \in \{1, \ldots, T\} \), there exists, for every \( j \in \{0, \ldots, k\} \), a unique \( t_j \in \{1, \ldots, T\} \) such that \( A_j = A^{t_j} \). Just as in the proof of Theorem 1, the chronology of choices \( \{A^{t_j}, a^{t_j}\}_{j=0}^{k} \) generates a cycle of revealed preference relations connecting any of its member to itself. One can therefore establish the required violation of \( T\text{-SARP} \) by just the same argument as in the proof of Theorem 1. Proving that a violation of \( T\text{-SARP} \) by the chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) implies
exists two (possibly identical) linear orderings that results from one change in preferences as per Definition 7. This means that there are two linear orderings on X and one period \( t \in \{1, ..., T\} \) such that \( a' \gtrless_1 a'_j \) for all \( a'_j \in A^j \) and \( j \in \{1, ..., T\} \) such that \( j < t \) and \( a' \gtrless_2 a'_v \) for all \( a'_v \in A^v \) and all \( v \in \{1, ..., T\} \) such that \( v \geq t \). If the two linear orderings are identical, then this means that the decision maker is behaving as if maximizing a single time-invariant ordering so that her chronological choice behaviour will satisfy T-SARP (and therefore trivially the requirement of Axiom 3). Assume now that the two linear orderings are distinct and, by contradiction, assume that this chronology of choices violates the requirement of T-SARP after 1 change. This amounts to assuming that there are periods \( r, s, t \) in the set \( \{1, ..., T\} \) satisfying \( r \leq s < t \) for which one has \( x \gtrsim^{rs} y \) and \( y \succ^{t'} x \) for some distinct \( x, y \in X \) and that there are also distinct \( w, z \in X \) for which one observes \( w \gtrsim^{uv} z \) and \( z \succ^\tau w \) for some periods \( u, v, \tau \) such that \( t' \leq u \leq v < \tau \). We first show that having both \( x \gtrsim^{rs} y \) and \( y \succ^{t'} x \) implies that \( r < t \leq t' \). By contradiction, suppose first that \( t \leq r < t' \). Since one has \( a^v \gtrless_2 a^v \) for all \( a^v \in A^v \) and all \( v \geq t \), the fact of observing both \( x \gtrsim^{rs} y \) and \( y \succ^{t'} x \) would imply, given the definition of \( \gtrsim^{rs} \) and \( \succ^{t'} \) and the transitivity of \( \gtrless_2 \), that both \( x \succ^{t} y \) and \( y \succ^{t} x \) hold, which is a contradiction. Similarly, if \( r < t' < t \), and given the fact that \( a^j \gtrless_1 a^j \) for all \( a^j \in A^j \) and all \( j \) such that \( j < t \), observing both \( x \gtrsim^{rs} y \) and \( y \succ^{t'} x \) would imply, given the definition of \( \gtrsim^{rs} \) and \( \succ^{t'} \) and the transitivity of \( \gtrless_1 \), that both \( x \succ^{t} y \) and \( y \succ^{t} x \) hold, which is also a contradiction. Since \( r < t' \leq t' \), one has that \( a^v \gtrless_2 a^v \) for all \( a^v \in A^v \) and all \( v \geq t \). But then, assuming the existence of \( w, z \in X \) for which one has \( w \gtrsim^{uv} z \) and \( z \succ^\tau w \) for some periods \( u, v, \tau \) such that \( t' \leq u \leq v < \tau \) would lead to the conclusion that both \( w \succ^{t} z \) and \( z \succ^{t} w \) hold, which is a contradiction. Hence a chronology of choices that results from one change in preferences satisfies T-SARP after 1 change.

In order to prove the converse implication, consider a chronology of choices \( \{A^i, a^i\}_{i=1}^T \) satisfying T-SARP after 1 change. If there are no \( r, s, t \in \{1, ..., T\} \) satisfying \( 1 \leq r < s < t \) for which one has \( x \gtrsim^{rs} y \) and \( y \succ^{t} x \) for some distinct \( x, y \in X \), then this means that the chronology of choices satisfies T-SARP. In that case, set \( \gtrless_1 = \gtrless \) where \( \gtrless \) is the linear ordering whose existence was established in Theorem 1 and let \( \gtrless_2 \) be any linear ordering whatsoever. As shown in Theorem 1, the linear ordering \( \gtrless \) will rationalize the chronology of choices from 1 up to \( T \).

Suppose now that there exists some periods \( r, s, t \) satisfying \( 1 \leq r < s < t \leq T \) for which one has \( x \gtrsim^{rs} y \) and \( y \succ^{t} x \) for some distinct \( x, y \in X \). Define then \( \hat{t} \) to be the smallest \( t \) for which \( x \gtrsim^{rs} y \) and \( y \succ^{t} x \) for some distinct \( x, y \in X \) and time periods \( 1 \leq r < s < t \leq T \). By definition of this \( \hat{t} \), the chronology of choices \( \{A^i, a^i\}_{i=1}^T \) satisfies T-SARP on the time horizon \( \{1, ..., \hat{t} - 1\} \). Since the chronology of choices satisfies T-SARP after 1 change, it also satisfies T-SARP on the
(non-empty) time horizon \( \{ \hat{t}, ..., T \} \). The result then follows from applying Theorem 1 to the time horizons \( \{ 1, ..., \hat{t} - 1 \} \) and \( \{ \hat{t}, ..., T \} \) sequentially. ■

A.1.4 Proof of Theorem 3

For the “if” part of the theorem, assume by contradiction that a chronology of choices \( \{ A^t, d^t \}_{t=1}^T \) results from the maximization of a discovered preference by one trial as per Definition 10 but that it violates T-SARP after 1 trial (Axiom 4). Hence, there are periods \( r, s \) and \( t \) such that \( r \leq s < t \) and some distinct \( x \) and \( y \in X \), for which one has \( x \succ_d r_s y \) and \( y \succ_t^l x \). By definition of \( x \succ_d^r y \), there is a sequence \( \{ t_j \}_{j=1}^k \) of \( k \) time periods (not necessarily ordered by time) in the set of time periods \( \{ r, r + 1, ..., s - 1, s \} \subset \{ 1, ..., T \} \) such that \( x = a^{i_1} \), \( a^{i_j} \succ_d^r a^{i_{j+1}} \) for all \( j = 1, ..., k - 1 \), \( y \in A^{i_k} \) and \( y = a^q \) for some period \( q < t_k \). By definition of \( a^{i_j} \succ_d^r a^{i_{j+1}} \) for all \( j = 1, ..., k - 1 \), there are, for any such \( j, \) periods \( r_j \) and \( s_j \) in \( \{ 1, ..., T \} \) satisfying \( r_j < t_j \) and \( s_j < t_j \) such that \( a^{i_j} = a^{t_j}, a^{i_{j+1}} \in A(t_j) \) and \( a^{i_{j+1}} = a^{s_j} \). Since the chronology of choices \( \{ A^t, d^t \}_{t=1}^T \) results from the maximization of a discovered preference by one trial, there exists a linear ordering \( \succeq \) on \( X \) such that \( x = a^{t_1} \succeq a^{t_2} \succeq ... \succeq a^{t_k} \). Since \( y \in A^{i_k} \) and \( y = a^q \) for some period \( q < t_k \), it follows from the maximization of a discovered preference by one trial that \( a^{t_k} \succeq y \). By the transitivity and the linearity of \( \succeq \) (as \( x \) and \( y \) are distinct) one has \( x \succ y \). But then, assuming \( y \succ y \) for \( r \leq s < t \) implies, under the assumption that the chronology of choices \( \{ A^t, d^t \}_{t=1}^T \) results from the maximization of a discovered preference by one trial as per Definition 10, that \( y \succ x \), which is a contradiction.

To prove the other implication, consider a chronology of choices \( \{ A^t, d^t \}_{t=1}^T \) that satisfies T-SARP after 1 trial and define the following “definite revealed preference” relation \( \succ_C^D \):

\[
x \succ_C^D y \iff \exists t \in \{ 1, ..., T \} \text{ such that } x \succ_d^t y
\]

(3)

Notice that this binary relation can be empty. This would happen, for example, for a chronology of choices in which the same menu is available at every period and the decision maker chooses that same alternative from that same menu at every period. In such a trivial case, the decision maker would never experience anything other than this chosen option, and there would be no pair of distinct alternatives that could be compared by \( \succ_C^D \). That is, the decision maker would never be given the opportunity to express any “definite preference”. In such a case, the choice behaviour can be (trivially) rationalized by any linear ordering \( \succeq \) whatsoever. Indeed, take any linear ordering \( \succeq \) and consider any period \( t \) for which \( x \succeq a^t \) for some \( x \in A^t \) and all \( a^t \in A^t \) but for which \( x \neq a^t \). There may not be any such \( t \), in which case the linear ordering \( \succeq \) rationalizes the choice behaviour in the usual sense. If however such a \( t \) exists, we then know from the emptiness of the binary relation \( \succ_C^D \) that either \( a^s \neq a^t \) for all \( s < t \) or \( x \neq a^s \) for all \( s < t \). Hence the chronological choice behaviour is trivially rationalized as resulting from the maximization of a discovered preference after one trial when \( \succ_C^D \) is empty. If \( \succ_C^D \) is not empty, one can define its transitive closure \( \succ_C^D \).
by:

\[ x \succsim^D_C y \iff \exists \{x_j\}_{j=0}^l \text{ for some } l \geq 1 \text{ such that:} \]

\[
\begin{align*}
x_0 &= x, \\
x_l &= y \text{ and,} \\
x_j \succsim^D_C x_{j+1} &\text{ for all } j = 0, \ldots, l - 1
\end{align*}
\]

Let us now show that the (transitive) binary relation \( \succsim^D_C \) is also antisymmetric. By contradiction, suppose there are two distinct \( x \) and \( y \in X \) such that \( x \succsim^D_C y \) and \( y \succsim^D_C x \).

By definition of \( \succsim^D_C \), there are two sequences of triples of periods \( \{r_j, s_j, t_j\}_{j=0}^l \) and \( \{r'_j, s'_j, t'_j\}_{j=0}^l \) (for some \( l \) and \( l' \geq 1 \)) satisfying, for every \( j \), \( r_j < s_j < t_j \), \( r'_j < t'_j \) and \( s'_j < t'_j \) for which one has:

\[
x_j = a^{r_j} = a^{s_j}, \quad x_{j+1} = a^{s_j} \quad \text{and} \quad x_j \in A^{r_j}
\]

as well as:

\[
x'_j = a^{r'_j} = a^{s'_j}, \quad x'_{j+1} = a^{s'_j} \quad \text{and} \quad x'_{j+1} \in A^{r'_j}
\]

for two sequences of alternatives \( \{x_j\}_{j=0}^l \) and \( \{x'_j\}_{j=0}^{l'} \) satisfying \( x_0 = x_0' = x \) and \( x_l = x'_{l} = y \). This generates a cycle of revealed definite preference connecting alternatives in \( X \) that can be initiated at every period of the sets of periods \( \{t_j\}_{j=0}^l \) and \( \{t'_j\}_{j=0}^{l'} \) defined above. In particular, one can take the maximal (with respect to the natural ordering of time) of these periods, and apply the reasoning of the proof of Theorem 1 to obtain the required violation of \( T\text{-SARP after 1 trial} \). Since \( \succsim^D_C \) is antisymmetric and transitive, it can be extended to a linear ordering \( \preceq \) using Spilrajn extension theorem.

Let us now show that the chronology of choices \( \{A^t, a^t\}_{t=1}^T \) results from the maximization of the discovered preference \( \preceq \) by one trial as per Definition 10. Consider any \( t \in \{1, \ldots, T\} \). Either \( a^t \succeq a'_t \) for all \( a'_t \in A^t \) or there exists some \( x \in A^t \) such that \( x \neq a^t \) and \( x \succeq a^t \). In the first case, \( \succsim \) rationalizes the choice made in the choice problem at \( t \) and there is nothing to prove. In the second case, take without loss of generality the alternative \( x \in A^t \) to be such that \( x \succeq a'_t \) for all \( a'_t \in A^t \). By assumption \( x \neq a^t \). Suppose that, contrary to the requirement that the chronology of choices \( \{A^t, a^t\}_{t=1}^T \) results from the maximization \( \preceq \) by one trial as per Definition 10, there exists a period \( r < t \) such that \( x = a^r \) and a period \( s < t \) such that \( a^t = a^s \). It then follows from the definition of \( \succsim^D_C \) that \( a^t \succsim^D_C x \) and, since \( \preceq \) is an extension of \( \succsim^D_C \), that \( a^t \succeq x \). This means that we have both \( x \succeq a^t \) and \( a^t \succeq x \), a contradiction of \( \preceq \) being antisymmetric. \( \blacksquare \)
A.2 Characterization of changing preferences model for k-changes of preferences

This appendix shows how the results of Section 4 can be extended to a decision maker who changes her preferences k times (for 0 ≤ k ≤ T − 1). We start by defining what it means for a chronology of choices to results from the choices of such a decision maker.

**Definition 11** A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) results from at most k changes in preferences, with 0 ≤ k ≤ T − 1, if there is a sequence of k + 1 linear orderings \( \{\succsim_j\}_{j=0}^{k} \) and a sequence of k periods \( \{t_j\}_{j=1}^{k} \) satisfying \( t_j \in \{2, ..., T\} \) for all j and \( t_1 \leq t_2 \leq ... \leq t_k \), such that, for any \( j = 1, ..., k \) and any period \( r \in \{t_{j-1}, t_{j-1} + 1, ..., t_j - 1\} \) (with the convention that \( t_0 = 1 \) and that the set of k periods is empty if \( k = 0 \)), one has \( a^r \succsim_{j-1} a'_r \) for all \( a^r \in A^r \) while for any period \( s \geq t_k \), one has \( a^s \succsim_k a'_s \) for all \( a'_s \in A^s \).

The following property characterizes a chronology of choices that results from at most k change in preferences.

**Axiom 5** (T-SARP after k changes) A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) satisfies T-SARP after k changes if the existence of a sequence of k triplets of periods \( \{q_j, r_j, s_j\}_{j=1}^{k} \) in the set \( \{1, ..., T\} \) satisfying \( s_{j-1} \leq q_j \leq r_j \leq s_j \) for all \( j \in \{1, ..., k\} \) (defining \( s_0 = 0 \)) and a sequence of k pairs of alternatives \( \{x_j, y_j\}_{j=1}^{k} \) in X for which one has \( x_j \succsim_{j-1} y_j \) and \( y_j \succsim_j x_j \) for \( j = 1, ..., k \) implies that one cannot have \( w \succsim_u z \) and \( z \succsim_v w \) for any two distinct alternatives \( w \) and \( z \in X \) and periods \( u, v \) and \( \tau \) such that \( s_k \leq u \leq v < \tau \leq T \).

Just like its one-preference change cousin, this property requires that if one observes k-consecutive violations of T-SARP taking place between period 1 and some period \( t_k \), then one cannot observe further violations of T-SARP occurring after \( t_k \).

**Theorem 4** A chronology of choices \( \{A^t, a^t\}_{t=1}^{T} \) satisfies T-SARP after k changes if and only if it results from at most k changes in preferences.

**Proof.** In one direction, suppose that \( \{A^t, a^t\}_{t=1}^{T} \) is a chronology of choices that violates T-SARP after k changes. Hence, using Axiom 5, there is a sequence of k triplets of periods \( \{q_j, r_j, s_j\}_{j=1}^{k} \) in \( \{1, ..., T\} \) satisfying \( s_{j-1} \leq q_j \leq r_j \leq s_j \) for all \( j \in \{1, ..., k\} \) (defining \( s_0 = 0 \)) and a sequence of k pairs of alternatives \( \{x_j, y_j\}_{j=1}^{k} \) in X for which one has \( x_j \succsim_{j-1} y_j \) and \( y_j \succsim_j x_j \) for \( j = 1, ..., k \) and there are also two distinct alternatives \( w \) and \( z \in X \) for which one observes \( w \succsim_u z \) and \( z \succsim_v w \) for periods \( u, v \) and \( \tau \) such that \( s_k \leq u \leq v < \tau \leq T \).

We first show that for any \( j \in \{1, ..., k\} \), there are two distinct linear orderings \( \succsim_{1j} \) and \( \succsim_{2j} \) on \( X \) and one period \( \rho_j \) satisfying \( q_j = \rho_j \leq s_j \) such that \( a^{q_j} \succsim_{1j} a_{\rho_j} \) for all \( a_{\rho_j} \in A^{\rho_j} \) for some period \( q_j \) satisfying \( q_j \leq \rho_j \). For this sake, let \( \succsim_{C}^{q_j} \) be the transitive closure of the (direct) revealed preference relation observed between periods \( q_j \) and \( r_j \). That is, \( x \succsim_{C}^{q_j} y \) holds.
for two distinct options \( x \) and \( y \) in \( X \) if and only if there is a sequence \( \{i^j_h\}_{h=0}^{\tilde{t}} \) of \( \tilde{t} + 1 \) time periods (not necessarily indexed by time) satisfying \( i^j_h \in \{q_j, ..., r_j\} \) for all \( h = 0, ..., \tilde{t} \) such that:

\[
\begin{align*}
  x &= a^{i^0_{i^j}}_1, \\
  a^{i^j}_{i^j} &\in A^{i^j-1} \text{ for } h = 1, ..., \tilde{t} \text{ and,} \\
  y &\in A^{i^j_0}.
\end{align*}
\]

Two cases must be considered:

(i) \( \preceq_{q_j r_j} \) is antisymmetric and

(ii) \( \succeq_{q_j r_j} \) is not antisymmetric.

In case (i), one can appeal to Spilrajn extension lemma to exhibit a linear ordering \( \succeq_{1 j} \) that extends the transitive and antisymmetric binary relation \( \preceq_{q_j r_j} \) and rationalizes all choices made between \( q_j \) and \( r_j \) in the sense that \( a^i \succeq_{1 j} a \) for all \( a \in A^1 \) for all \( i \in \{1, ..., T\} \) such that \( q_j \leq i \leq r_j \). If this is the case, then the fact of having \( x_j \succeq_{q_j r_j} y_j \) (and therefore \( x_j \succeq_{1 j} y_j \)) and \( y_j \succ_{s_j} x_j \) implies that the ordering rationalizing the choice made at period \( s_j \) is different from \( \succeq_{1 j} \). We then take \( r_j = p_j \) and we define in this case \( \preceq_{s_j} \) to be any transitive and linear completion of the revealed preference binary relation \( \succ_{s_j} \). Suppose now that case (ii), holds. Then there are two distinct alternatives \( x \) and \( y \in X \) for which both \( x \succeq_{q_j r_j} y \) and \( y \succeq_{q_j r_j} x \) hold. This means that:

\[
\exists\{i^j_h\}_{h=0}^{\tilde{t}} \text{ with } i^j_h \in \{1, ..., T\} \text{ for } h = 0, ..., \tilde{t} \text{ (with } \tilde{t} \geq 0) \text{ and } q_j \leq i^j_h \leq r_j \text{ such that:}
\]

\[
\begin{align*}
  (i) &\quad x = a^{i^0_{i^j}}_1, \\
  (ii) &\quad a^{i^j}_{i^j} \in A^{i^j-1} \text{ for } h = 1, ..., \tilde{t}, \\
  (iii) &\quad y \in A^{i^j_0}.
\end{align*}
\]

and

\[
\exists\{i^j_h\}_{h=0}^{\tilde{t}'} \text{ with } i^j_h \in \{1, ..., T\} \text{ for } h = 0, ..., \tilde{t}' \text{ (with } \tilde{t}' \geq 0) \text{ and } q_j \leq i^j_h \leq r_j \text{ such that:}
\]

\[
\begin{align*}
  (i) &\quad y = a^{i^0_{i^j}}_1, \\
  (ii) &\quad a^{i^j}_{i^j} \in A^{i^j-1} \text{ for } h = 1, ..., \tilde{t}', \\
  (iii) &\quad x \in A^{i^j_0}.
\end{align*}
\]

Just like in the proof of Theorem 1, consider the sets of time periods \( T = \bigcup_{h=0}^{\tilde{t}} \{i^j_h\} \)

and \( T' = \bigcup_{h=0}^{\tilde{t}'} \{i^j_h\} \) involved in expressions (4) and (5) respectively. Since these two expressions define a cycle of revealed preference relations connecting alternative \( x \) (or any other chosen option in the cycle) to itself, this cycle can be started at any \( h \in T \cup T' \) that one wishes. In particular, \( h \) can be chosen to be the maximal (with respect
to the natural ordering of the time periods underlying the set \( \{1, ..., t\} \) such period in \( T \cup T' \). We then have \( a^h \succ^h a^\sigma \) for some period \( \sigma < h \leq r^j \). By definition of the cycle induced by expressions (4) and (5), there are also periods \( \rho \) and \( \rho' \in (T \cup T') \setminus \{h\} \) satisfying \( q^j \leq \rho \leq \rho' < h \leq r^j \) that produce the opposite indirect preference \( a^\sigma \succ^\rho a^h \). Define in this case the binary relation \( \succ_C \) to be the transitive closure of the (direct) revealed preference relation observed between periods \( \rho \) and \( \rho' \). If this transitive closure is antisymmetric, then apply the reasoning used in case (i) above by defining \( \succ_C \) to be a linear ordering extension (thanks to Spera lemma) of \( \succ_C \). In this case, one has \( a^{\delta_1} \succ_1 a_{s_j} \) for all \( a_{s_j} \in A^{\delta_1} \) for all periods \( g_j \) satisfying \( \rho \leq g_j \leq \rho' \), and we can set \( \rho_j = h \) by defining \( \succ_C \) to be any linear extension of the direct revealed preference relation \( \succ^h \). If the transitive closure \( \succ_C \) is not antisymmetric, then redo the reasoning leading to the existence of a revealed preference cycle described by expressions similar to (4) and (5) but applied this time to \( \succ_C \). We remember that the set of periods between \( \rho \) and \( \rho' \) is a strict subset of the set of periods between \( q^j \) and \( r^j \) (period \( h \) is not in the set of periods between \( \rho \) and \( \rho' \)). We then reapply the same reasoning to the revealed preference cycle associated to \( \succ_C \) by isolating a (late) period \( h' \) that reveals a direct preference for some alternative \( x \) over some alternative \( y \) while succeeding to a pair of time periods \( \rho'' \) and \( \rho''' \) satisfying \( q^j \leq \rho'' < h' < h \leq r^j \) for which \( y \succ x \). We observe that every time we resort to such a generation of a revealed preference cycle due to the failure of transitive closure of the indirect revealed preference of the preceding step to be antisymmetric, we do so on a set of periods that is strictly smaller (at least by one period) than the set of the preceding step. Since the number of periods is finite, this iterative procedure therefore eventually ends. It ends precisely when the transitive closure of the relevant revealed preference relation is indeed antisymmetric, as per case (i). In this case, we will be able to establish the existence of the two distinct linear orderings \( \succ_{1j} \) and \( \succ_{2j} \) on \( X \) and the period \( \rho_j \) by using the reasoning above. If the transitive closure is not antisymmetric, then we redo again the procedure that generates a new preference cycle of the type induced by expressions (4) and (5). We observe that every time we resort to such a generation of a revealed preference cycle due to the failure of transitive closure of the indirect revealed preference of the preceding step to be antisymmetric, we do so on a set of periods that is strictly smaller (at least by one period) than the set of the preceding step. Since the number of periods is finite, this iterative procedure therefore eventually ends. It ends precisely when the transitive closure of the relevant revealed preference relation is indeed antisymmetric, as per case (i). In this case, we will be able to establish the existence of the two distinct linear orderings \( \succ_{1j} \) and \( \succ_{2j} \) on \( X \) and the period \( \rho_j \) before which \( \succ_{1j} \) is used and at which \( \succ_{2j} \) is used.

We have therefore established the existence of a sequence of \( k \) pairs of distinct linear orderings \( \{\succ_{1j}, \succ_{2j}\}_{j=1}^{k} \) and periods \( \{\rho_j\}_{j=1}^{k} \) satisfying, for all \( j = 1, ..., k \), \( q_j < \rho_j \leq s_j \) and \( a^{\delta_j} \succ_{1j} a_{s_j} \) for all \( a_{s_j} \in A^{\delta_j} \) for some period \( g_j \) such that \( q_j \leq g_j < \rho_j \) and \( a^\rho_j \succ_{2j} a_{s_j} \) for all \( a_{s_j} \in A^\rho \). Of course one can not exclude the possibility that, for some \( j \in \{2, ..., k\} \), one could have \( \succ_{2j-1} = \succ_{1j} \) (e.g. the second linear ordering of the time interval \( j-1 \) is the same than the first ordering of the time interval \( j \)). If this arises at every period \( j = 1, ..., k-1 \), then one would have at least \( 2k-(k-1) = k+1 \) distinct linear orderings that would rationalize choices made between period \( q_1 \) and period \( s_k \). We write "at least" because we can not rule out the possibility that there are additional linear orderings that could rationalize choices made at periods other than \( g_j \) or \( \rho_j \) (for \( j = 1, ..., k \)). However, if such additional linear orderings are needed
to rationalize choices made between $q_1$ and $s_k$, then this means that the chronology of choices $\{A^i, a^i\}_{i=1}^T$ do not result from at most $k$ changes in preferences as per Definition 11 and the “if” part of the proof is complete. Assume therefore that exactly $k + 1$ linear orderings $(\succeq_{11}, \succeq_{21}, \succeq_{22}, \ldots, \succeq_{2k-1}, \succeq_{1k}, \succeq_{2k})$ (satisfying $\succeq_{11} \neq \succeq_{21}$ and $\succeq_{2j-1} \neq \succeq_{1j}$ for all $j = 2, \ldots, k$) are required for rationalizing choices between $q_1$ and $s_k$. We can without loss of generality assume that the ordering $\succeq_{11}$ rationalizes also the choices made before period $q_1$ (that is $a^t \succeq_{11} a^s$ for all $a^t \in A^t$ and all periods $t$ such that $1 \leq t \leq q_1$). Indeed, assuming that $\succeq_{11}$ does not rationalize the choices made before period $q_1$ would mean that another ordering would be required to perform this rationalization. And again, this would mean that at least $k + 2$ orderings would be required to rationalize choices made between periods 1 and $s_k$, and this would complete the “if” part of the proof. Hence, we have just established the existence of $k$ periods $p_1, \ldots, p_k$ satisfying, for all $j = 1, \ldots, k$, $q_j \leq y_j < p_j \leq s_j$ and such that $a^{p_j} \succeq_{2j-1} a^{y_j}$ for all $a^{y_j} \in A^{y_j}$ and all periods $l_j$ satisfying $p_j \leq l_j < p_j$ (for $j = 1, \ldots, k$) and with the convention that $p_0 = 1$ and $\succeq_{20} = \succeq_{11}$) and $a^{m} \succeq_{2k} a^{m}$ for all $a^{m} \in A^{m}$ for all periods $m$ such that $p_k \leq m \leq s_k$. Since $\{A^i, a^i\}_{i=1}^T$ is a chronology of choices that violates Axiom 5, we know that there are also two distinct alternatives $w$ and $z \in X$ for which one observes $w \succeq_{zz} z$ and $z \succ w$ for periods $u, v$ and $\tau$ such that $s_k \leq u < v < \tau \leq T$. Either the ordering $\succeq_{2k}$ that rationalizes choices between periods $p_k$ and $s_k$ also rationalizes the choices made from period $s_k$ to period $v$ or it does not. If it does not, then the “if” part of the proof is complete because it indicates that at least $k + 2$ pairwise distinct orderings are required to rationalize the chronology of choices $\{A^i, a^i\}_{i=1}^T$, in violation of it resulting from at most $k$ changes in preferences as per Definition 11. If on the other hand the ordering $\succeq_{2k}$ rationalizes the choices made between periods $s_k$ and $v$, then it follows from the transitivity of $\succeq_{2k}$ that $w \succeq_{2k} z$. But then, the fact that $z \succ w$ shows that there is an ordering compatible with $\succ$ and distinct from $\succeq_{2k}$ that rationalizes the choices made between some periods $p$ and $p'$ satisfying $v < p \leq \tau \leq p'$. This again implies that there are at least $k + 2$ successively distinct linear orderings that rationalize the chronology of choices $\{A^i, a^i\}_{i=1}^T$, in violation of it resulting from at most $k$ changes in preferences as per Definition 11. This completes the “if” part of the proof.

To prove the “only if” part of the proof, consider a chronology of choices $\{A^i, a^i\}_{i=1}^T$ satisfying T-SARP after $k$ changes. Define recursively the periods $s_i^* \in \{1, \ldots, T\}$ (for $i = 1, \ldots, j^*$ for some integer $j^* \in \{1, \ldots, T\}$) by the existence of periods $q_i^*$ and $r_i^* \in \{1, \ldots, T\}$ satisfying $s_i^* - 1 \leq q_i^* \leq r_i^* < s_i^*$ (setting $s_0^* = 1$) such that:

$$x_i \succ q_i^*, r_i^*; y_i \text{ and } y_i \succ s_i^* x_i$$

(6)

for some distinct alternatives $x_i$ and $y_i \in X$ and by the absence of periods $s \in \{1, \ldots, T\}$ satisfying $s_i^* - 1 \leq s < s_i^*$ for which one can find periods $q, r \in \{s_{i-1}, \ldots, s - 1\}$ such that:

$$q \leq r, x \succeq_{yy} y \text{ and } y \succ^s x \text{ for some distinct } x \text{ and } y \in X$$

(7)

In words, $s_i^*$ is the first period where the direct revealed preference is observed to contradict a previously observed indirect revealed preference, $s_i^*{2}$ is the first period after $s_i^*$ where the direct revealed preference contradicts a previously observed indirect revealed
preferences occurring after $s^*_t$, and so on. It is of course possible that $j^* = 0$, in which case there are no period $s^*_t$ where the direct revealed preference relation $\succ^*_t$ contradicts a previously observed indirect preference relation. If this is the case, then the chronology of choice satisfies T-SARP, and can therefore be rationalized by one single linear ordering by virtue of Theorem 1. It then clearly results by at most $k$ (in fact 0) changes in preferences as per Definition 11. Suppose now that $0 < j^* \leq k$. In this case, for all $i = 1, \ldots, j^*$, define $\succ^*_t \subseteq C$ as the transitive closure of the direct revealed preference relation observed between periods $s^*_i - 1$ and $s^*_i - 1$ (setting $s^*_0 = 1$). By definition of $s^*_i$, this transitive binary relation is antisymmetric. Hence by Spilrajn lemma, it can be extended to a complete linear ordering $\succ^*_t \subseteq C$ that rationalizes the choices made between $s^*_i - 1$ and $s^*_i - 1$ in the sense that $a^\sigma \succ^*_t a^\sigma$ for all $a^\sigma \in A^\sigma$ and all $\sigma \in \{s^*_i - 1, \ldots, s^*_i - 1\}$. Since expression (4) holds for some distinct alternatives $x_i$ and $y_i \in X^\sigma$ and some periods $q^*_i$ and $r^*_i \in \{1, \ldots, T\}$ satisfying $s^*_i - 1 \leq q^*_i \leq r^*_i < s^*_i$, we must have $\succ^*_t \subseteq C$ for all $i = 1, \ldots, j^* \leq k$. This means that the chronology of choices results from at most $j^* \leq k$ changes in preference. Since T-SARP after $k$ changes (Axiom 5) requires that $j^* \leq k$, this completes the proof. ■
A.3 Characterization of preference discovery model for $k$-trials

This appendix characterizes the revealed preference discovery model for any $k$ number of trials (with $1 \leq k \leq T - 1$). In order to do so, we first redefine what is meant by direct definitely revealed preferred in this context.

**Definition 12** Given a chronology of choices $\{A_t, a_t\}_{t=1}^T$ and any period $t \in \{1, \ldots, T\}$ and distinct alternatives $x$ and $y \in X$, we say that $x$ is directly revealed definitely preferred to $y$ at $t$ after $k$ trials, denoted $x \succeq_{k,y} y$, if and only if there are $k$ distinct periods $r_1, \ldots, r_k$ and $k$ distinct periods $s_1, \ldots, s_k$ satisfying $r_h < t$ and $s_h < t$ for $h = 1, \ldots, k$ such that:

(i) $x = a^{r_h}$ and, $y = a^{s_h}$ for all $h = 1, \ldots, k$

(ii) $x = a^t$ and $y \in A^t$.

In other words, the chronology of choices directly reveals a definite preference for $x$ over $y$ at $t$ after $k$ trials if $x$ is chosen over $y$ at a period $t$ that follows $k$ choices of $x$ and $k$ choices of $y$. The indirect revealed definitely preferred after $k$ trials relation can analogously be defined as follows.

**Definition 13** Given a chronology of choices $\{A_t, a_t\}_{t=1}^T$ and any periods $r$ and $s$ such that $r \leq s$ and any distinct alternatives $x$ and $y \in X$, we say that $x$ is indirectly revealed definitely preferred to $y$ between periods $r$ and $s$ after $k$ trials, denoted $x \succeq_{r,s} y$, if and only if there is a sequence $\{t_i\}_{i=1}^j$ of $j$ time periods ($j \geq 1$) in the set $\{r, r+1, \ldots, s-1, s\}$ for which one has:

(i) $x = a^{t_1}$ and there are $k$ distinct periods $r_1, \ldots, r_k \in \{1, \ldots, T\}$ satisfying $r_h < t_1$ and $x = a^{r_h}$ for all $h = 1, \ldots, k$.

(ii) $a^{r_h} \succeq_{t_h} a^{r_{h+1}}$ for all $h = 1, \ldots, k-1$ (if any) and,

(iii) $y \in A^{t_j}$ and there are $k$ distinct periods $s_1, \ldots, s_k \in \{1, \ldots, T\}$ satisfying $s_h < t_j$ and $y = a^{s_h}$ for all $h = 1, \ldots, k$.

Given these definitions, we now formally define what we mean for a chronology of choices to result from the maximization of a discovered preference after $k$ trials.

**Definition 14** A chronology of choices $\{A_t, a_t\}_{t=1}^T$ results from the maximization of a discovered preference after $k$ trials if there exists a linear ordering $\succeq$ on $X$ such that, for all $t \in \{1, \ldots, T\}$, either:

(i) $a^t \succeq a_t$ for all $a_t \in A^t$ for which there exist $k$ distinct periods $s_1, \ldots, s_k$, all preceding $t$, such that $a_h = a^{s_h}$ for all $h = 1, \ldots, k$,

(ii) There are no $k$ distinct periods $s_1, \ldots, s_k$, all preceding $t$, such that $a^t = a^{s_h}$ for all $h = 1, \ldots, k$.

That is, a chronology of choices results from the maximization of a discovered preference after $k$ trials if there exists a preference such that the choice made by the decision maker at every period is either the “best” option for that preference among all alternatives that have been previously tried $k$ times or the chosen option has not itself been tried $k$ times in the past.

It is immediate to see that the following property characterizes this behaviour.
**Axiom 6 (T-SARP after k trials)** A chronology of choices \( \{A^t, a^t\}_{t=1}^T \) satisfies T-SARP after k trials if for any periods \( r, s \) and \( t \) such that \( r \leq s < t \) and some distinct \( x \) and \( y \in X \), one cannot have \( x \gtrsim_{rs} y \) and \( y \succ_{td} x \).

This axiom is nothing else than the T-SARP axiom applied to the revealed definite preference after \( k \)-trials. It says that we should never observe a violation of T-SARP for two alternatives that have been previously chosen \( k \) times in the past.

**Theorem 5** A chronology of choices \( \{A^t, a^t\}_{t=1}^T \) satisfies T-SARP after \( k \) trials if and only if it results from the maximization of a discovered preference after \( k \) trials.

**Proof.** The argument proceeds as in the Proof of Theorem 3, but by applying the reasoning to the binary relations \( \gtrsim_{rs} \) and \( \gtrsim_{td} \) (instead of \( \gtrsim_{td} \) and \( \gtrsim_{rs} \)) and to the appropriate definition of the revealed definite preference after-\( k \) trials \( \gtrsim_{Dk} \) (as in expression (3) applied to the binary relation \( \gtrsim_{Dk} \)). □